LBL-12704
Preprint

Submitted to Nuclear Physics

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January 1982

Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

LBL-12704

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25 January, 1982

Abstract

A theory of isospin asymmetric matter and of neutron star matter in particular, that is compatible with bulk nuclear properties, is formulated in the framework of a relativistic field theory and solved in the mean field approximation. The self-consistent conditions defining the sigma, omega, pion and rho fields are derived for the normal and the pion condensed states in explicit form for asymmetric matter, which is in chemical equilibrium, and with prescribed total charge.

This work was supported by the Director, Office of Energy Research, Division of Nuclear Physics of the Office of High Energy and Nuclear Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098.

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Normal and Pion Condensed States in Neutron Star Matter in a Relativistic Field Theory Constrained by Bulk Nuclear Properties I

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Contents

I. Introduction

This paper formulates a relativistic field theory of dense isospin asymmetric matter. The solutions are developed in the mean field approximation and relativistic covariance is retained throughout. The theory by construction is sufficiently rich that it can be constrained by the four important bulk properties of nuclear matter, 1) saturation binding, 2) saturation density, 3) compressibility, and 4) the symmetry energy. In addition the constraints of fixed charge and stability with respect to chemical composition are imposed.

The theory is constrained by the known bulk properties of matter to insure a more reliable extrapolation to higher densities. The special application that we have in mind is neutron star matter, which is extremely asymmetric but not pure neutron.

In addition to the normal state, we study the pion condensed state, which previously we studied in symmetric nuclear matter.[1] The isospin asymmetry introduces additional theoretical and technical problems not present in the description of symmetric matter. The above set of constraints have never before been imposed on a field theoretic model of neutron star matter in the pion condensed state. Earlier work on field theories of dense matter and references can be found in several reviews.[2]

The plan of the paper is as follows. In section II a Lagrangian is constructed based on interacting nucleons and mesons in the various spin-isospin channels. The starting point is the Lagrangian that has been used extensively by Walecka,[3] which incorporates the isoscalar mesons σ and ω_μ. To this is added interacting fields representing the isovector pion and the rho meson. Special care has to be taken so as to avoid introducing an instability to arbitrarily growing fields. In section III the mean field approximation is introduced and the self-consistency equations for the mean meson fields are derived. The stress-energy tensor is calculated in section IV, and the stability of the mean field approximation to our Lagrangian is confirmed. Various conserved currents are constructed in section V. In section VI we discuss a number of properties of the pion condensed state and the system in general that is described by our Lagrangian. In section VII we derive the quasi-particle eigenvalues and propagator and show how to construct the nucleon source currents that appear in the field equations, energy density and pressure, in terms of momentum integrals over the Fermi volume. Finally, in section VIII the solution of the mean field equations are summarized.

II. Choice of Lagrangian

It is now widely believed that the fundamental description of matter is a relativistic field theory of interacting quarks and gluons. Nevertheless, at nuclear matter density this substructure does not play a dominant role. Presumably, there is a critical density above which it does become important. However, at moderate density and momentum transfer, the exchange quanta giving rise to the interactions between the nucleons in nuclear matter can be represented by the exchange of mesons. In this regime, nuclear matter can be described through a field theory of interacting nucleons and mesons in the various spin-isospin channels, (J^{π},I), such as

$$\sigma(0^{+},0), \quad \omega_{\mu}(1^{-},0), \quad \underset{\sim}{\pi}(0^{-},1), \quad \underset{\sim}{\rho}_{\mu}(1^{-},1)$$

and the properties of matter can be characterized by the expectation values of the various nucleon current operators

$$J_{\Gamma}(x) = \langle\bar{\psi}(x)\Gamma\psi(x)\rangle \tag{1a}$$

where Γ is one of the operators,

$$\Gamma = \left\{1,\gamma_{\mu},\gamma_{5},\gamma_{\mu}\gamma_{5},\sigma_{\mu\nu}\right\} \times \left\{1,\underset{\sim}{\tau}\right\} \tag{1b}$$

In the normal ground state of spin-isospin symmetric matter, only $\Gamma = 1$ and γ_0 have nonvanishing expectation values. In isospin asymmetric matter, the current corresponding to $\gamma_{\mu}\underset{\sim}{\tau}$ will also be finite in the ground state. Above normal density additional currents may acquire a nonvanishing expectation value. The so-called pion condensate state corresponding to

$$\underset{\sim}{J}_{\mu 5} = \langle\bar{\psi}\,\gamma_{\mu}\gamma_{5}\underset{\sim}{\tau}\psi\rangle \quad \text{or} \quad \underset{\sim}{J}_{5} = \langle\bar{\psi}\gamma_{5}\underset{\sim}{\tau}\psi\rangle \tag{2}$$

are examples that we focus on in this paper.

We wish here to extend an earlier work[1] to asymmetric matter, with special attention to neutron star matter. The matter of neutron stars is unlikely to be pure neutron. Energetically, it is favorable for a small admixture of protons and electrons to be present, as an estimate based on free Fermi gases readily indicates, which estimate is strengthened by more refined calculation.[4] Therefore, we do not idealize this neutron rich matter as pure neutron but treat the complexity of general asymmetric matter, and the pion-rho condensate thereof.

In symmetric matter the nuclear matter constraints are the saturation density and binding and the compression modulus which lies in the range 200-300 MeV.[5] For asymmetric matter the neutron-proton asymmetry energy is an additional constraint that we impose, which will be used to determine the effective ρ-N coupling constant. For the density-temperature regime in which the nucleons do not dissolve into quark matter, the Lagrangian of matter has the general form

$$\mathcal{L} = \mathcal{L}^{free} + \mathcal{L}^{int} \tag{3}$$

where $\mathcal{L}^{free}$ is the sum of the free Lagrangian for nucleon, σ, ω, π, and ρ mesons and $\mathcal{L}^{int}$ is a suitable interaction Lagrangian, which we now discuss.

A theory based only on the isoscalar mesons σ and ω_μ was considered very early in nuclear physics[6], and interest in this theory has revived in recent years through the work of Walecka[3] and his collaborators and Boguta.[7] The meson fields are Yukawa coupled to the nucleon fields. The attraction of the σ-meson and repulsion of the vector ω-meson are responsible for the binding of symmetric nuclear matter at the saturation density. Boguta and Bodmer added self-interactions of the scalar field that can be exploited to account for the compressibility of nuclear matter. This theory, which is constrained by those three bulk properties, is then able to give an excellent account of many single-particle properties of finite nuclei, including the charge density, spin-orbit interaction, single-particle states, and energy dependence of the optical potential.[7] Therefore, for these fields we take the interaction Lagrangian

$$\mathcal{L}^{int}_{\sigma,\omega,N} = g_\sigma \sigma(\bar{\psi}\psi) - g_\omega \omega_\mu (\bar{\psi} \gamma^\mu \psi) - U(\sigma) \tag{4a}$$

(The notation and conventions of Bjorken and Drell[8] are used throughout this work). The last term is the σ-self-interaction, which is taken from ref. 7 in the form

$$U(\sigma) = (\frac{1}{3} bm + \frac{1}{4} cg_\sigma\sigma)(g_\sigma\sigma)^3 \tag{4b}$$

with m the nucleon mass and b and c dimensionless constants.

The π-N interaction is known to be attractive in the p-state, whereas the pseudoscalar coupling

$$-f_\pi \ \underset{\sim}{\pi}\cdot (\bar{\psi}\gamma_5 \underset{\sim}{\tau}\psi) \tag{5}$$

is dominated by a large repulsive s-wave interaction. For this reason, both here and in our earlier work, we choose instead the pseudovector coupling

$$\mathcal{L}_{\pi N} = - g_\pi (\partial_\mu \underset{\sim}{\pi})\cdot(\bar{\psi} \gamma^5\gamma^\mu \underset{\sim}{\tau} \psi) \tag{6}$$

In our work on symmetric matter we ignored the ρ-meson altogether. The complete justification is given à posteriori later in this work. The time-like component ρ^0 is clearly not relevant. The vector-isovector field has as its source the current $\langle\bar{\psi}\gamma^0\underset{\sim}{\tau}\psi\rangle$, which vanishes in isospin symmetric matter. The isospin 3-component, for example, is the difference in the proton and neutron densities.

On the other hand, in asymmetric matter the (isovector) rho meson makes a very important contribution to the symmetry energy. In its absence the symmetry energy in the normal state would arise only from the difference in contribution to the kinetic energy of protons and neutrons, which is insufficient to account for the observed asymmetry coefficient. However, care must be taken in constructing the rho meson interactions. The rho meson should be coupled to the conserved isospin current. Otherwise, instabilities may occur in the theory.

We recall now the general method, based on Noether's theorem, for constructing the currents of a given Lagrangian. If the Lagrangian, a function of the fields ϕ_i and their space-time derivatives $\partial_\mu \phi_i$, is invariant to an infinitesimal transformation of the fields according to

$$\phi_i \rightarrow \phi_i + \Lambda_j \, F_{ij} \, (\phi_1 \ldots \phi_j \ldots) \tag{7}$$

then the system possesses a continuous internal symmetry and

$$J_i^{\,\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)} \, F_{ji} \tag{8}$$

are a set of conserved four-currents corresponding to that symmetry, for which

$$\partial_\mu J_i^{\,\mu} = 0 \tag{9}$$

For the isospin rotation, the transformation is

$$\psi \rightarrow \left(1 - \frac{i}{2} \, \underset{\sim}{\Lambda} \cdot \underset{\sim}{\tau}\right) \psi \qquad \text{, spinor field} \tag{10}$$

$$\underset{\sim}{\phi} \rightarrow \underset{\sim}{\phi} + \underset{\sim}{\Lambda} \times \underset{\sim}{\phi} \qquad \text{, isovector field} \tag{11}$$

where $\underset{\sim}{\Lambda}$ is an arbitrary vector of infinitesimal length in isospin space. Therefore, we find three conserved currents corresponding to the three directions in isospin space that are given by

$$\underset{\sim}{J}^{\mu} = \frac{1}{2}\bar{\psi}\,\gamma^{\mu}\underset{\sim}{\tau}\psi + \sum_{r} \underset{\sim}{\phi}_r \times \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\underset{\sim}{\phi}_r)} \tag{12}$$

where the sum is over all isovector fields $(\underset{\sim}{\pi},\underset{\sim}{\rho})$ or particularly

$$\underset{\sim}{J}^{\mu} = \frac{1}{2}\bar{\psi}\,\gamma^{\mu}\underset{\sim}{\tau}\psi + \underset{\sim}{\pi} \times \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\underset{\sim}{\pi})} + \underset{\sim}{\rho}_{\nu} \times \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\underset{\sim}{\rho}_{\nu})} \tag{13}$$

Now we introduce the rho meson interactions by coupling $\underset{\sim}{\rho}^{\mu}$ to the total isospin current. Two steps are involved in finding this current. First construct $\underset{\sim}{J}'_{\mu}$, corresponding to the Lagrangian $\mathcal{L}'$ described by (3), (4), (6). Then consider

$$\mathcal{L} = \mathcal{L}' - \underset{\sim}{\rho}^{\mu} \cdot \underset{\sim}{J}'_{\mu} \tag{14}$$

Because of the derivative terms in $\underset{\sim}{J}'_{\mu}$ arising from the free Lagrangians of π and ρ in $\mathcal{L}'$, the last term in (14) will generate additional contributions to the current. So these too must be included. The resulting Lagrangian is

$$\begin{aligned} \mathcal{L} = {} & \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\sigma} + \mathcal{L}_{\omega} + \mathcal{L}_{\pi} + \mathcal{L}_{\rho} \\ & + g_{\sigma}\,\sigma\,(\bar{\psi}\psi) - g_{\omega}\,\omega_{\mu}\,(\bar{\psi}\,\gamma^{\mu}\psi) - U(\sigma) \\ & - g_{\pi}(\partial_{\mu}\,\underset{\sim}{\pi})\cdot(\bar{\psi}\gamma_5\gamma^{\mu}\underset{\sim}{\tau}\psi) - g_{\rho}\,\underset{\sim}{\rho}_{\mu}\cdot\underset{\sim}{J}^{\mu} \end{aligned} \tag{15a}$$

where $\underset{\sim}{J}^{\mu}$ are the three total isospin currents corresponding to $\mathcal{L}$ (one for each isospin axis). They are

$$\begin{aligned} \underset{\sim}{J}^{\mu} = {} & \frac{1}{2}(\bar{\psi}\gamma^{\mu}\,\underset{\sim}{\tau}\psi) + \underset{\sim}{\pi} \times \partial^{\mu}\underset{\sim}{\pi} + \underset{\sim}{\rho}_{\nu} \times \underset{\sim}{\rho}^{\mu\nu} \\ & + g_{\pi}\,(\bar{\psi}\,\gamma_5\gamma^{\mu}\underset{\sim}{\tau} \times \underset{\sim}{\pi}\psi) + g_{\rho}(\underset{\sim}{\rho}^{\mu} \times \underset{\sim}{\pi}) \times \underset{\sim}{\pi} \\ & + 2g_{\rho}(\underset{\sim}{\rho}^{\mu} \times \underset{\sim}{\rho}^{\nu}) \times \underset{\sim}{\rho}_{\nu} \end{aligned} \tag{15b}$$

and the free field Lagrangians are

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\partial\!\!\!/ - m)\,\psi \tag{15c}$$

$$\mathcal{L}_{\sigma} = \frac{1}{2}(\partial_{\mu}\sigma\partial^{\mu}\sigma - m_{\sigma}^{2}\sigma^{2}) \tag{15d}$$

$$\mathcal{L}_{\omega} = -\frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2}m_{\omega}^{2}\omega_{\mu}\omega^{\mu} \tag{15e}$$

$$\mathcal{L}_{\pi} = \frac{1}{2}(\partial_{\mu}\underset{\sim}{\pi}\cdot\partial^{\mu}\underset{\sim}{\pi} - m_{\pi}^{2}\,\underset{\sim}{\pi}\cdot\underset{\sim}{\pi}) \tag{15f}$$

$$\mathcal{L}_{\rho} = -\frac{1}{4}\,\underset{\sim}{\rho}_{\mu\nu}\cdot\underset{\sim}{\rho}^{\mu\nu} + \frac{1}{2}m_{\rho}^{2}\,\underset{\sim}{\rho}_{\mu}\cdot\underset{\sim}{\rho}^{\mu} \tag{15g}$$

where

$$\omega_{\mu\nu} = \partial_{\mu}\omega_{\nu} - \partial_{\nu}\omega_{\mu}\quad,\quad \underset{\sim}{\rho}_{\mu\nu} = \partial_{\mu}\underset{\sim}{\rho}_{\nu} - \partial_{\nu}\underset{\sim}{\rho}_{\mu} \tag{15h}$$

The signs of the free field Lagrangians are chosen so as to yield a positive definite Hamiltonian density.

III. Self-Consistency Equations for the Mean Fields

The rather complicated Lagrangian (15) will be used to describe symmetric and asymmetric nuclear matter at densities up to some moderate multiple of the saturation density. By construction, it can and will be constrained to yield the four important bulk properties of nuclear matter mentioned earlier. As the full set of coupled equations is intractable, we seek a solution in the mean field (Hartree) approximation. With this approximation, (15) is to be regarded as an effective many-body Lagrangian with coupling constants determined by many-body properties and not by hadron scattering data. A critique of this approximation was given in Ref. 1. It consists of solving the Dirac equation derived from (15) in which all meson fields are replaced by ground state expectation values. The nucleon ground state of the system is then constructed as a degenerate Hartree state out of these solutions. The nucleon source currents that appear in the meson field equations, and which couple them one to the other, are replaced by expectation values in the above nucleon ground state, and the field equations are solved for the mean meson fields self-consistently with these sources. From now on, the meson field symbols will denote these mean values.

The Dirac equation that follows from $\mathcal{L}$ is

$$\begin{aligned}(i\not\partial - g_\omega \not\omega - (m - g_\sigma \sigma) - g_\pi \gamma_5 \gamma^\mu \underset{\sim}{\tau}\cdot\partial_\mu \underset{\sim}{\pi} - \frac{1}{2} g_\rho \gamma^\mu \underset{\sim}{\tau}\cdot\underset{\sim}{\rho}_\mu \\ - g_\rho g_\pi \gamma_5 \gamma^\mu \underset{\sim}{\tau} \times \underset{\sim}{\pi}\cdot\underset{\sim}{\rho}_\mu)\psi = 0\end{aligned} \tag{1}$$

In infinite homogeneous matter, the sources for the σ and ω fields, $\langle\bar{\psi}(x)\ \psi\ (x)\rangle$ and $\langle\bar{\psi}(x)\gamma^\mu\psi(x)\rangle$ are independent of x, so the scalar and vector fields are constants. However, because the π-N interaction has its first important contribution in the p-wave, the pion field, if finite, will vary in space-time to exploit the interaction (the derivative coupling in $\mathcal{L}$).

Therefore, we investigate pion fields that are oscillating in space-time of the form

$$\underset{\sim}{\pi}(x) = \bar{\pi}\,(\underset{\sim}{u} \cos kx + \underset{\sim}{v} \times \underset{\sim}{u} \sin kx) \tag{2}$$

where $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are orthogonal unit vectors in isospin space. We will later specialize to a particular choice

$$u = (1,0,0), \quad v = (0,0,1) \tag{3}$$

in which case

$$\pi_\pm(x) = \frac{1}{\sqrt{2}}\,\bar{\pi} e^{\pm ikx} \quad , \quad \pi_0 = 0 \tag{4}$$

$$kx = k_0 t - \underset{\sim}{k}\cdot\underset{\sim}{x} \tag{5}$$

As in ref. 1 this field can be written in terms of a space-time dependent rotation operator in isospin space about the direction $\underset{\sim}{v}$.

$$R_v = \exp\left(-\frac{1}{2}\, i(kx)\ \underset{\sim}{\tau}\cdot\underset{\sim}{v}\right) \tag{6}$$

as

$$\underset{\sim}{\tau}\cdot\underset{\sim}{\pi}(x) = \bar{\pi} R_v\ \underset{\sim}{\tau}\cdot\underset{\sim}{u}\ R_v^{+} \tag{7a}$$

and

$$\underset{\sim}{\tau}\cdot\partial_\mu\ \underset{\sim}{\pi}\ (x) = \bar{\pi}\ k_\mu\ R_v\ \underset{\sim}{\tau}\cdot\underset{\sim}{v} \times \underset{\sim}{u}\ R_v^{+} \tag{7b}$$

Although the Dirac equation (1) depends on this space-time dependent field, it is readily seen that a new quasi-particle field defined by

$$\psi_v(x) = R_v^{+}\psi(x) \tag{8}$$

obeys a Dirac equation in which the space-time dependence of the π field has been removed. It is

$$\Big[i\partial\!\!\!/ - g_\omega \omega\!\!\!/ - (m - g_\sigma \sigma) + k\!\!\!/ \tau\cdot(\tfrac{1}{2} \underset{\sim}{v} + g_\pi \overline{\pi}\, \gamma_5\, \underset{\sim}{v} \times \underset{\sim}{u})$$

$$- \tfrac{1}{2}\, g_\rho \gamma^\mu\, R_v^+ \underset{\sim}{\tau}\cdot \rho_\mu\, R_v - g_\rho g_\pi\, \gamma_5 \gamma^\mu\, R_v^+ \underset{\sim}{\tau}\cdot\underset{\sim}{\pi} \times \underset{\sim}{\rho}_\mu\, R_v \Big] \psi_v = 0 \qquad (9)$$

To eliminate the space-time dependence acquired by the last two terms as a result of this transformation, two choices are possible. The first is that $\underset{\sim}{\rho}_\mu$ is a space-time constant and perpendicular to the $\underset{\sim}{\pi}$ field, i.e.,

$$\underset{\sim}{\rho}_\mu = \rho_\mu\, \underset{\sim}{v} \qquad (10)$$

The second is that $\underset{\sim}{\rho}_\mu$ vary in space-time analogous to the pion field. However, the kinetic energy associated with this is unfavored by our Lagrangian. For (10), the Dirac equation for the quasi-particle fields becomes

$$\Big[i\partial\!\!\!/ - g_\omega \omega\!\!\!/ - (m - g_\sigma \sigma) + (k\!\!\!/ - g_\rho\, \rho\!\!\!/)\ \underset{\sim}{\tau}\cdot(\tfrac{1}{2} \underset{\sim}{v} + g_\pi \overline{\pi} \gamma_5\, \underset{\sim}{v} \times \underset{\sim}{u})\Big] \psi_v = 0 \qquad (11)$$

Now since all terms are space-time constants, the transformed fields can be chosen as eigenstates of momentum

$$\psi_v = U(p) e^{-ipx} \qquad (12)$$

where the eight-component spinor, U(p), satisfies

$$[P\!\!\!/ - m^* + K\!\!\!/\ \underset{\sim}{\tau}\cdot(\tfrac{1}{2} \underset{\sim}{v} + g_\pi\, \overline{\pi}\gamma_5\, \underset{\sim}{v} \times \underset{\sim}{u})]\ U(p) = 0 \qquad (13)$$

Here we have defined shifted momenta and an effective mass according to

$$P_\mu = p_\mu - g_\omega\, \omega_\mu \qquad (14a)$$

$$K_\mu = k_\mu - g_\rho\, \rho_\mu \qquad (14b)$$

$$m^* = m - g_\sigma\, \sigma \qquad (14c)$$

This equation has the same appearance as Eq. 24 of ref. 1 for the Dirac field in symmetric matter. All that has changed is that k is replaced by K. Therefore, the Dirac propagator, which is the inverse of the operator in (13), has the same form as that determined in our earlier work, with the above substitution. Likewise the eigenvalue equation has the same form as before.

We now wish to derive the self-consistency conditions on the mean values of the fields that are imposed by the Euler-Lagrange equations derived from $\mathcal{L}$ (15) with (2,3,10). After some considerable algebra, these are found to be

$$m_\sigma^2 \sigma = g_\sigma \langle \bar{\psi}_V \, \psi_V \rangle - \frac{dU}{d\sigma} \tag{15a}$$

$$m_\omega^2 \omega_\mu = g_\omega \langle \bar{\psi}_V \, \gamma_\mu \psi_V \rangle \tag{15b}$$

$$[-K_\mu K^\mu - g_\rho^2 \rho_\mu \rho^\mu + m_\pi^2] \, \bar{\pi} = -g_\pi \langle \bar{\psi}_V \, \gamma_5 \, \not{K} \, \tau_2 \, \psi_V \rangle \tag{15c}$$

$$[m_\rho^2 + (g_\rho \bar{\pi})^2] \, \rho^\mu = g_\rho \, J_3{}^\mu \tag{15d}$$

The bracket < > denotes the ground state expectation value with respect to the Hartree state constructed from the lowest energy eigenstates of (13). The expectation value of the 3rd isospin four-current in (II-15b), written now in terms of the quasi-particle operators ψ_V, is

$$J_3^\mu = \bar{\pi}^2 \, K^\mu + \langle \bar{\psi}_V \, \gamma^\mu \, (\tfrac{1}{2} \tau_3 + g_\pi \bar{\pi} \, \gamma_5 \, \tau_2) \psi_V \rangle \tag{16}$$

For the ansatz (4), (10) the other two isospin components of $\underset{\sim}{J}^\mu$ vanish as will be evident later. Therefore, our ansatz is consistent with $\mathcal{L}$. Equations (13) - (16) comprise the self-consistency equations for the mean values of the fields. They have acquired a deceptively simple appearance through our choice of notation. All the fields are coupled to one another in these transcendental equations, through the nucleon source currents and the shifted pion momentum K^{μ}, except the ω_o field, which is given directly by the baryon density ρ_B.

$$m_{\omega}^{2}\omega_{o} = g_{\omega} \langle\psi_{v}^{+}\psi_{v}\rangle = g_{\omega} \rho_{B} \tag{17}$$

In deriving the above and some later results, useful relations between the $\underset{\sim}{\pi}$ and $\underset{\sim}{\rho}$ fields (2) and (10) are

$$\partial_{\mu}\underset{\sim}{\pi} = k_{\mu} \underset{\sim}{v} \times \underset{\sim}{\pi} \tag{18a}$$

$$\underset{\sim}{\pi} \times \partial_{\mu}\underset{\sim}{\pi} = \bar{\pi}^{2} k_{\mu} \underset{\sim}{v} \tag{18b}$$

$$\partial_{\mu} \underset{\sim}{\pi} \times \underset{\sim}{v} = k_{\mu} \underset{\sim}{\pi} \tag{18c}$$

$$\partial_{\mu}\underset{\sim}{\pi} \times \underset{\sim}{\rho}^{\mu} = k_{\mu} \rho^{\mu} \underset{\sim}{\pi} \tag{18d}$$

From (18a) and (7b) it also follows that

$$R^{+}(\underset{\sim}{\pi} \times \underset{\sim}{\tau}) \cdot \underset{\sim}{v} R = \bar{\pi} \underset{\sim}{\tau}\cdot\underset{\sim}{v} \times \underset{\sim}{u} \tag{19}$$

This is used in the transformation of the term

$$\langle\bar{\psi} \gamma^{\mu} \gamma_{5} (\underset{\sim}{\pi} \times \underset{\sim}{\tau})\cdot \underset{\sim}{v} \psi\rangle = \pi \langle\bar{\psi}_{v} \gamma^{\mu} \gamma_{5} \underset{\sim}{\tau}\cdot\underset{\sim}{v} \times \underset{\sim}{u} \psi_{v}\rangle \tag{20}$$

IV. Stress-Energy Tensor

The energy density and pressures are given by the diagonal components of the stress-energy tensor. Its canonical form is[8]

$$\mathcal{T}_{\mu\nu} = -g_{\mu\nu}\,\mathcal{L} + \sum \frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi)}\,\partial_{\nu}\phi \tag{1}$$

where ϕ denotes one of the fields ψ, σ, ω.... and the sum is over the fields. Thus

$$\mathcal{T}_{\mu\nu} + g_{\mu\nu}\,\mathcal{L} = (\bar{\psi}\gamma_{\mu} i\partial_{\nu}\psi) + [\partial_{\mu}\underset{\sim}{\pi} - g_{\pi}\,(\bar{\psi}\gamma_5\,\gamma_{\mu}\underset{\sim}{\tau}\psi) - g_{\rho}\underset{\sim}{\rho}_{\mu}\,\times\underset{\sim}{\pi}]\cdot\partial_{\nu}\underset{\sim}{\pi} \tag{2}$$

Taking the ground state expectation value of this, and expressing the nucleon currents in terms of the tranformed Dirac fields (III-8) we find an interesting structure for the stress-energy tensor, namely,

$$\bar{\mathcal{T}}_{\mu\nu} + g_{\mu\nu}\,\bar{\mathcal{L}} = \langle\bar{\psi}_V\,\gamma_{\mu}\,p_{\nu}\psi_V\rangle + J_{3\mu}\,k_{\nu} \tag{3}$$

where $J_{3\mu}$ is one of the three isospin currents (III-16), corresponding to the isospin 3-axis. While not manifestly symmetric, we will prove later that $\mathcal{T}_{\mu\nu}$ is in fact diagonal. The ground state expectation value of our Lagrangian (II-15), after exploiting the Dirac equation, is given by

$$\bar{\mathcal{L}} = \langle\mathcal{L}\rangle = -\frac{1}{2}m_{\sigma}^2\sigma^2 - U(\sigma) + \frac{1}{2}m_{\omega}^2\omega_{\mu}\omega^{\mu} + \frac{1}{2}\bar{\pi}^2\,(k_{\mu}k^{\mu} - m_{\pi}^2)$$

$$+ \frac{1}{2}m_{\rho}^2\rho_{\mu}\rho^{\mu} - g_{\rho}\,\bar{\pi}^2\,\rho_{\mu}\,K^{\mu} \tag{4}$$

The energy density is given by $\bar{\mathcal{T}}_{00}$, namely,

$$\varepsilon = -\bar{\mathcal{L}} + \langle\psi_V\,\gamma_0\,p_0\,\psi_V\rangle + J_{3o}\,k_o \tag{5}$$

We return now to the discussion initiated in section II concerning the stability of the system. Combining the $k_{\mu}k^{\mu}$ and $\rho_{\mu}K^{\mu}$ in $\bar{\mathcal{L}}$, we find after a little algebra

$$-\mathcal{L} = \frac{1}{2} m_\sigma^2\sigma^2 + U(\sigma) - \frac{1}{2} m_\omega^2\omega_\mu\omega^\mu + \frac{1}{2} m_\pi^2\bar{\pi}^2$$

$$- \frac{1}{2} m_\rho^2 \rho_\mu \rho^\mu - \frac{1}{2} \bar{\pi}^2 [K_\mu K^\mu + g_\rho^2\rho_\mu\rho^\mu] \qquad (6)$$

The space-like parts of the four-vector products are all positive definite, which prevents all space-like components of the fields from growing large. The time-like components have negative signs, but this is of no consequence, since they are connected to conserved quantities. As noted in (III-17), ω_0 is determined by the baryon number and from (III-15d), ρ_0 is determined by the total isospin. Finally, the $J_{30}k_0$ term in ε together with $-\frac{1}{2}\bar{\pi}^2K_0^2$ yields a positive contribution $\frac{1}{2}\bar{\pi}^2K_0^2$. Thus the system described in the mean field approximation by the Lagrangian (II-15) possesses stable finite field configurations.

V. Conserved Currents

Noether's theorem and the construction of conserved currents was given in section II by equations (7-9). We have already constructed the three isospin currents. The ground state expectation of the third of these currents was given in (III-16). The other two could be obtained from (II-15b), but for our ansatz for the π and ρ fields, they vanish identically. Now we construct two other relevant currents, the baryon current and the total charge current.

i) Baryon Current

Under the transformation affecting baryons only

$$\psi \to (1 + i\Lambda)\ \psi\ , \tag{1}$$

we find the baryon current whose ground state expectation is

$$j_B^\mu = \langle\bar{\psi}\ \gamma^\mu\psi\rangle = \langle\bar{\psi}_V\gamma^\mu\psi_V\rangle \tag{2}$$

The time-like component is the baryon density $\rho_B = j_B^0$. The conserved baryon number is

$$N = \int j_B^o d^3x = \int \langle\psi^+\psi\rangle d^3x \tag{3}$$

ii) Charge Current

Under the transformation of all charged fields defined by

$$\psi \to \left(1 - i\Lambda\frac{1+\tau_3}{2}\right)\psi \tag{4a}$$

$$\underset{\sim}{\pi} \to \underset{\sim}{\pi} + \underset{\sim}{\Lambda} \times \underset{\sim}{\pi} \tag{4b}$$

$$\underset{\sim}{\rho}^\mu \to \underset{\sim}{\rho}^\mu + \underset{\sim}{\Lambda} \times \underset{\sim}{\rho}^\mu \tag{4c}$$

$$\underset{\sim}{\Lambda} = (0,0,1)\Lambda \tag{4d}$$

we calculate the charge current

$$j^\mu = \langle\bar{\psi}\gamma^\mu \frac{1+\tau_3}{2}\psi\rangle + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\pi_i)} \quad \varepsilon^{ijk}\pi_k\,\delta_{j3}$$

$$= \langle\bar{\psi}\gamma^\mu \frac{1+\tau_3}{2}\psi\rangle + (\underset{\sim}{\pi} \times \partial^\mu\underset{\sim}{\pi})_3 + g_\rho[(\underset{\sim}{\rho}^\mu \times \underset{\sim}{\pi}) \times \underset{\sim}{\pi}]_3 + g_\pi\langle\bar{\psi}\gamma_5\gamma^\mu(\underset{\sim}{\tau} \times \underset{\sim}{\pi})_3\psi\rangle \quad (5)$$

Transforming to the quasi-particle fields, and using (III-18, 20)

$$j^\mu = \langle\bar{\psi}_v\gamma^\mu \frac{1+\tau_3}{2}\psi_v\rangle + g_\pi\bar{\pi}\,\langle\bar{\psi}_v\,\gamma^\mu\gamma_5\tau_2\psi_v\rangle + \bar{\pi}^2\,K^\mu \quad (6)$$

Referring to the isospin current (III-16), we find, as expected,

$$j^\mu = J_3^\mu + \frac{1}{2}\,j_B^\mu \quad (7)$$

whose time-like component is

$$q = J_3^0 + \frac{1}{2}\,\rho_B \quad (8)$$

where q denotes the charge density and ρ_B the baryon density.

VI. Some Properties of the Pion Condensed State

As has been remarked elsewhere,[9] if pion-like excitations occur in the ground state, being bosons, they will macroscopically occupy the lowest available mode. Such a state corresponds to a coherent excitation of the pion field in the medium for which the ground state expectation of the pion field $<\underset{\sim}{\pi}(x)>$ does not vanish. These expectation values are the order parameters of the pion condensed phase. This phase is one of broken symmetry, since the pion field, being pseudoscalar, would vanish in an eigenstate of parity. Thus the phase transition corresponds to a structural change in the ground state, from a uniform isotropic mixture of neutrons and protons, to an aligned spin-isospin lattice of wave number $|\underset{\sim}{k}|$ and frequency k_o. The lattice structure was illustrated for symmetric matter in Ref. 1.

From our calculation of the charge current in section V, we can identify the contribution of the pion field to the charge. The pion density is

$$\rho_\pi = - \left(\frac{\partial\mathcal{L}}{\partial(\partial_o\underset{\sim}{\pi})} \times \underset{\sim}{\pi}\right)_3 = -(\underset{\sim}{p}_\pi \times \underset{\sim}{\pi})_3 = -i\ (p_\pi \pi^\dagger - \pi p_\pi{}^\dagger) \tag{1}$$

where $\pi \equiv \pi_+$, $\pi^\dagger \equiv \pi_-$, and the canonical momentum of the classical field, $\underset{\sim}{\pi}$, is denoted by

$$\underset{\sim}{p}_\pi = < \frac{\partial\mathcal{L}}{\partial(\partial_o\underset{\sim}{\pi})} > = \frac{\partial\overline{\mathcal{L}}}{\partial(\partial_o\underset{\sim}{\pi})} = \partial_o\underset{\sim}{\pi} - g_\rho \underset{\sim}{\rho}_o \times \underset{\sim}{\pi} + g_\pi <\bar{\psi}\ \gamma_o\ \gamma_5\ \underset{\sim}{\tau}\ \psi> \tag{2}$$

The Hamiltonian density H is the time-like diagonal component of $\mathcal{T}_{\mu\nu}$. by construction it is a function of the generalized coordinates and momenta, whereas $\mathcal{L}$ is a function of the coordinates and their space-time derivatives.

$$H = -\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_o\underset{\sim}{\pi})} \cdot \partial_o\underset{\sim}{\pi} + \frac{\partial\mathcal{L}}{\partial(\partial_o\psi)}\ \partial_o\psi \tag{3a}$$

or

$$\varepsilon = \overline{H} = \langle H \rangle = -\mathcal{L} + \underset{\sim}{p}_\pi \cdot \dot{\underset{\sim}{\pi}} + \langle \overline{\psi}\, i\gamma_o \partial_o \psi \rangle$$

$$= -\mathcal{L} + p_\pi \dot{\pi}^\dagger + \dot{\pi} p_\pi^\dagger + \langle \overline{\psi}\, i\gamma_o \partial_o \psi \rangle \qquad (3b)$$

Under an arbitrary variation of the pion field the energy must satisfy the relation

$$\delta\overline{H} = \mu_\pi\, \delta\, \rho_\pi \qquad (4)$$

which defines the chemical potential μ_π. Computing the right side from (1) and the left side using Hamilton's equations we have

$$\delta\overline{H} = \dot{\underset{\sim}{\pi}} \cdot \delta \underset{\sim}{p}_\pi - \dot{\underset{\sim}{p}}_\pi \cdot \delta \underset{\sim}{\pi} = \dot{\pi}^\dagger \delta p_\pi - \dot{p}_\pi^\dagger\, \delta\pi + cc \qquad (5)$$

$$= i\mu_\pi(\pi^\dagger \delta p_\pi - p_\pi^\dagger \delta\pi) + cc$$

Hence

$$\dot{\pi} = -\, i\mu_\pi \pi \quad , \dot{p}_\pi = -i\mu p_\pi \qquad (6)$$

shows that the frequency of the pion condensate is given by the chemical potential μ_π, as was first shown by Baym and Flowers.[10]

We see that (in accord with our convention III-(4)),

$$k_o = -\mu_\pi \qquad (7)$$

The mean pion density ρ_π can be calculated using (1,2) and (III-20),

$$\rho_\pi = \overline{\pi}^2\, K_o + g_\pi \overline{\pi} \langle \overline{\psi}_v\, \gamma_o\, \gamma_5\, \tau_2\, \psi_v \rangle \qquad (8)$$

As remarked already, pions, being bosons, can macroscopically occupy the lowest mode available in the medium. To determine the wave number of this mode we must solve the equation

$$\frac{\partial \overline{H}}{\partial \underset{\sim}{k}} = 0 \qquad (9)$$

where the variation is such that the baryon and charge densities are held fixed, and the fields satisfy the field equations (III-15). From the foregoing expressions, or referring to (IV-5) we have, after expressing the nucleon currents in terms of the quasi-particle fields,

$$\overline{H} = -\overline{\mathcal{L}} + \langle \overline{\psi}_v \, \gamma_o \, p_o \, \psi_v \rangle + J_{3o} k_o \tag{10}$$

where J_{3o} is the isospin density corresponding to the isospin four-current (III-16). According to (V-8) it is fixed by the baryon and charge densities. A variation of $\overline{H}$ subject to the above-mentioned constraints yields therefore

$$\begin{aligned} 0 &= \frac{\delta \overline{H}}{\delta \underset{\sim}{k}} = \frac{\partial \overline{H}}{\partial \underset{\sim}{k}} + \frac{\partial \overline{H}}{\partial \sigma} \frac{\delta \sigma}{\delta \underset{\sim}{k}} + \frac{\partial \overline{H}}{\partial \overline{\overline{\pi}}} \frac{\delta \overline{\pi}}{\delta \underset{\sim}{k}} + \ldots \\ &= \frac{\partial \overline{H}}{\partial \underset{\sim}{k}} = - \frac{\partial \overline{\mathcal{L}}}{\partial \underset{\sim}{k}} + \frac{\partial}{\partial \underset{\sim}{k}} \langle \overline{\psi}_v \, \gamma_o \, p_o \, \psi_v \rangle \end{aligned} \tag{11}$$

where, at constant baryon and charge density, and by virtue of the field equations

$$\frac{\partial \overline{H}}{\partial \sigma} = \frac{\partial \overline{H}}{\partial \overline{\overline{\pi}}} = \ldots = 0 \tag{12}$$

The first term on the right of (11) can be evaluated from (IV-4). The second can be evaluated by noting that p_o can be replaced through the Dirac equation (III-13) by the Dirac Hamiltonian for the quasi-particles,

$$H_D = \gamma_o \left[\underset{\sim}{\gamma} \cdot \underset{\sim}{p} + g_\omega \not{\omega} + m^* - \not{k} \, \underset{\sim}{\tau} \cdot \left(\frac{1}{2} \underset{\sim}{v} + g_\pi \overline{\pi} \, \gamma_5 \, \underset{\sim}{v} \times \underset{\sim}{u} \right) \right] \tag{13a}$$

Let the sum of the occupied Fermion quasi-particle eigenvalues be denoted by

$$\varepsilon_D = \langle \psi_v^+ \, H_D \, \psi_v \rangle \tag{13b}$$

Then

$$\frac{\partial}{\partial \underset{\sim}{k}} \langle \bar{\psi}_V \gamma_0 p_0 \psi_V \rangle = \langle \psi_V^\dagger \frac{\partial H_D}{\partial \underset{\sim}{k}} \psi_V \rangle + \varepsilon_D \frac{\partial}{\partial \underset{\sim}{k}} \langle \psi_V^\dagger \psi_V \rangle \tag{14}$$

The last term vanishes at constant baryon density. Hence (11) yields

$$\bar{\pi}^2 \underset{\sim}{K} + \langle \bar{\psi}_V \underset{\sim}{\gamma} (\frac{1}{2} \tau_3 + g_\pi \bar{\pi} \gamma_5 \tau_2) \psi_V \rangle = 0 \tag{15}$$

as the condition on $\underset{\sim}{k}$ that yields the lowest pion mode of the medium (consistent with the ansatz that the condensate is a plane wave).

According to (III-16), an immediate consequence of (15) is that the space part of the isospin current vanishes. This result, which is here derived explicitly from the condition (9), is a particular expression of a general theorem,[9] that the ground state expectation of those three-currents corresponding to conserved quantities must vanish. In this case, it is the total isospin that is conserved. The vanishing of the spatial part of the isospin current tells us through (III-15d) that

$$\rho^i = 0 \quad (i = 1,2,3) \tag{16}$$

Another condition that we must impose, to find the ground state, is

$$\frac{\delta \bar{H}}{\delta k_0} = 0 \tag{17}$$

We have seen already that k_0 is connected to the pion chemical potential, and the above condition imposes chemical equilibrium. Condition (17) yields the identity contained in the $\mu = 0$ component of (III-16). This is not devoid of content because through (V-8) it reads

$$\bar{\pi}^2 K_0 + \langle \bar{\psi}_V \gamma_0 (\frac{1}{2} \tau_3 + g_\pi \bar{\pi} \gamma_5 \tau_2) \psi_V \rangle = q - \frac{1}{2} \rho_B \tag{18}$$

where q and ρ_B are the externally chosen charge and baryon density.

The theorem, quoted above, could be invoked to assert the vanishing of the baryon current and the total charge current. However, it is also instructive to see explicit proofs. Let us denote the eigenvalues of the Dirac equation (III-13) by $P_0 = E_\kappa(\underset{\sim}{p})$ where κ is a label characterizing the spin-isospin. In the next section we will construct explicitly the eigenvalue equation, but it is not needed at this point. From the Dirac equation (III-13) we can write for a single-particle state characterized by $\underset{\sim}{p}$, κ, for which we denote the creation operator by $a^\dagger_{p\kappa}$,

$$\langle\psi_V^\dagger H_D \psi_V\rangle_{\underset{\sim}{p}\kappa} \equiv \langle 0|a_{\underset{\sim}{p}\kappa} \psi_V^\dagger H_D \psi_V a^\dagger_{\underset{\sim}{p}\kappa}|0\rangle = E_\kappa(\underset{\sim}{p}) + g_\omega\omega_0 \tag{19}$$

where we recall (III-14a) in writing the connection $p_0 = P_0 + g_\omega\omega_0$. Take a derivative with respect to ω_0 to get the normalization condition

$$\langle\psi_V^\dagger \psi_V\rangle_{\underset{\sim}{p}\kappa} = 1 \tag{20}$$

Next take a derivative with respect to a component of momentum p_i, to obtain

$$\langle\bar{\psi}_V \gamma_i \psi_V\rangle_{\underset{\sim}{p}\kappa} = \frac{\partial}{\partial p_i} E_\kappa(\underset{\sim}{p}) \tag{21}$$

Hence our ground state expectation value of γ_i is

$$\langle\bar{\psi}_V \gamma_i \psi_V\rangle = \sum_\kappa \int \frac{d^3p}{(2\pi)^3}\left(\frac{\partial}{\partial p_i} E_\kappa(\underset{\sim}{p})\right)\theta_\kappa$$

$$\sum_\kappa \int \frac{dp_j\, dp_k}{(2\pi)^3} \int dE_\kappa\, \theta_\kappa = 0 \tag{22}$$

where

$$\theta_\kappa = \theta[E_F - E_\kappa(\underset{\sim}{\rho})] \tag{23a}$$

$$\theta(x) = 1,\ x > 0, \qquad \theta(x) = 0, \qquad x < 0 \tag{23b}$$

The integration is over the occupied momentum region up to the Fermi energy, E_F. The integral vanishes because of course $E_\kappa(\underset{\sim}{p})$ is equal to the Fermi energy everywhere on this surface. Hence, we have proven, according to (V-2) that the spatial components of j_B^μ vanish. The vanishing of the baryon current in the ground state also tells through (V-7) that the total charge current vanishes, since we have already proven that the total isospin current vanishes. A further consequence of the vanishing of the baryon current is

$$\omega_k \equiv 0 \quad (k = 1,2,3) \tag{24}$$

as follows from (III-15b).

There is yet another consequence of the vanishing isospin current. The diagonal space components of the stress-energy tensor (IV-3) assume the form

$$\mathcal{T}_{ii} = \mathcal{L} + \langle\bar{\psi}_V \gamma_i p_i \psi_V\rangle \tag{25}$$

More than this, all three components can be proven equal (isotropy of pressure in the ground state) as now shown. Using (21) we can write

$$\begin{aligned}
\langle\bar{\psi}_V \gamma_i p_i \psi_V\rangle &= \sum_\kappa \int \frac{d^3p}{(2\pi)^3} p_i \left(\frac{\partial}{\partial p_i} E_\kappa (\underset{\sim}{p})\right) \theta_\kappa \\
&= \sum_\kappa \int \frac{d^3p}{(2\pi)^3} \left[\frac{\partial}{\partial p_i} (p_i E_\kappa) - E_\kappa\right] \theta_\kappa \\
&= \sum_\kappa \left[E_F \int \frac{dp_j dp_k}{(2\pi)^3} (p_{i>} - p_{i<})\theta_\kappa - \int \frac{d^3p}{(2\pi)^3} E_\kappa \theta_\kappa \right]
\end{aligned} \tag{26}$$

where $p_{i>}$ and $p_{i<}$ are the upper and lower values of p_i on the momentum surface located at $p_j p_k$. On this surface E_κ equals the Fermi energy E_F, which then can be taken out of the integral as a common factor. The difference $(p_{i>} - p_{i<})$ can be converted back to an integral on p_i so that

$$\langle\bar{\psi}_v \gamma_i p_i \psi_v\rangle = \sum_\kappa \int \frac{d^3p}{(2\pi)^3} (E_F - E_\kappa) \quad \theta_\kappa$$

$$= E_F \rho_B - \sum_\kappa \int \frac{d^3p}{(2\pi)^3} E_\kappa \quad \theta_\kappa \tag{27}$$

where ρ_B denotes the baryon density. The right side, being independent of the index i, tells us through (25) that all three diagonal space components of $\bar{\mathcal{T}}_{\mu\nu}$ are equal, and they are the pressure,

$$P = \bar{\mathcal{L}} + E_F \rho_B - \sum_\kappa \int \frac{d^3p}{(2\pi)^3} E_\kappa(\underset{\sim}{p}) \quad \theta_\kappa \tag{28}$$

provided that the stress-energy tensor is diagonal. We now prove this to be the case. First examine

$$\mathcal{T}_{io} = \langle\bar{\psi}_v \gamma_i p_o \psi_v\rangle + J_{3i} k_o \tag{29}$$

From (15) the last term vanishes. From (21) the first is

$$\langle\bar{\psi}_v \gamma_i p_o \psi_v\rangle = \sum_\kappa \int \frac{d^3p}{(2\pi)^3} \frac{\partial E_\kappa}{\partial p_i} (E_\kappa + g_\omega \omega_o) \theta_\kappa$$

$$= \sum_\kappa \int \frac{d^3p}{(2\pi)^3} \frac{\partial}{\partial p_i} \left(\frac{1}{2} E_\kappa^2 + g_\omega \omega_o E_\kappa\right) \theta_\kappa$$

$$= 0 \tag{30}$$

The integral vanishes because E_κ is equal to the Fermi energy on the surface bounding the region of integration. This proves that $\mathcal{T}_{io}$ vanishes. The proof that $\mathcal{T}_{oi}$ vanishes is more involved. We first use (30) and replace p_o by H_D since our ground state is an eigenfunction of H_D.

$$0 = \langle\bar{\psi}_v \gamma_i H_D \psi_v\rangle = \langle\bar{\psi}_v [\gamma_o p_i + (K_i \gamma_o - K_o \gamma_i)(\tfrac{1}{2} \tau_3 + g_\pi \bar{\pi} \gamma_5 \tau_2)] \psi_v\rangle \tag{31}$$

By recalling the expression for the isospin current (III-16), this relation can be written

$$<\overline{\psi}_v \gamma_o p_i \psi_v> = K_o (J_{3i} - \overline{\pi}^2 K_i) - K_i (J_{3o} - \overline{\pi}^2 K_o) = - J_{3o} k_i \tag{32}$$

where in the last line we have used (15) and (16). Therefore

$$\mathcal{J}_{oi} \equiv <\overline{\psi}_v \gamma_o p_i \psi_v> + J_{3o} k_i = 0 \tag{33}$$

There remain only the off-diagonal space components,

$$\begin{aligned} \mathcal{J}_{ij} &= <\overline{\psi}_v \gamma_i p_j \psi_v> + J_{3i} k_j \\ & \sum_\kappa \int \frac{d^3p}{(2\pi)^3} \frac{\partial E_\kappa}{\partial p_i} p_j \theta_\kappa \\ &= \sum_\kappa \int \frac{p_j dp_j dp_k}{(2\pi)^3} \int dE_\kappa \theta_\kappa = 0 \end{aligned} \tag{34}$$

where again (21) was used. This completes the proof that our stress-energy tensor has the diagonal form

$$\mathcal{J}_{\mu\nu} = \begin{pmatrix} \varepsilon & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix} \tag{35}$$

VII. Nucleon Eigenvalues, Propagator and Source Currents

The eigenvalues of the Dirac equation can be found by rationalizing the Dirac operator. For the field free case, $(\not{p} - m)u = 0$, this is accomplished by multiplying by $(\not{p} + m)$ yielding $p_\mu p^\mu - m^2 = 0$. Because of the complicated operator structure in our case (III.13), a fourth order equation is obtained instead, namely

$$D(P) \equiv ((PP) - \varepsilon_0^2)^2 - (PK)^2 - (2g_\pi \bar{\pi})^2\left[(PK)^2 - m^{*2}\,(KK)\right] = 0 \qquad (1)$$

where

$$\varepsilon_0^2 = m^{*2} - \left[\frac{1}{4} + (g_\pi \bar{\pi})^2\right](KK) \qquad (2)$$

and

$$(ab) = a_0 b_0 - \underset{\sim}{a}\cdot\underset{\sim}{b} \qquad (3)$$

Solving $D(P) = 0$ for P_0 yields the eigenvalue spectrum

$$p_0 = P_0 + g_\omega \omega_0 = \omega(\underset{\sim}{p}) \qquad (4)$$

Denote the four roots, P_0, by $E_\pm$ for the particle states and $\bar{E}_\pm$ for the anti-particle states so that

$$D(P) = [P_0 - E_-(\underset{\sim}{P})][P_0 - E_+(\underset{\sim}{P})][P_0 - \bar{E}_-(\underset{\sim}{P})][P_0 - \bar{E}_+(\underset{\sim}{P})] \qquad (5)$$

The four-fold spin-isospin degeneracy of free nucleons is partially lifted through the interactions. The + and - levels are doubly degenerate.

The nucleon propagator for the transformed Fermions can be written

$$\begin{aligned} S(P) = \frac{1}{D(P)} \Big\{ &(PP) - \varepsilon_0^2 - 2(PK)\left[\frac{1}{2}\underset{\sim}{\tau}\cdot\underset{\sim}{v} + g_\pi \bar{\pi}\,\gamma_5\,\underset{\sim}{\tau}\cdot\underset{\sim}{v} \times \underset{\sim}{u}\right] \\ &- 2g_\pi \bar{\pi}\, m^* \not{K}\,\gamma_5\,\underset{\sim}{\tau}\cdot\underset{\sim}{v} \times \underset{\sim}{u}\Big\} \Big\{ \not{P} + m^* + \not{K}\left[\frac{1}{2}\underset{\sim}{\tau}\cdot\underset{\sim}{v} - g_\pi \bar{\pi}\,\gamma_5\,\underset{\sim}{\tau}\cdot\underset{\sim}{v} \times \underset{\sim}{u}\right]\Big\} \\ \equiv N/D \end{aligned} \qquad (6)$$

From these results we can now state how to calculate the source currents.

According to Appendix A of Ref. 1, the expectation of an operator Γ in the transformed Dirac space can be calculated from

$$\langle\bar{\psi}_v \Gamma \psi_v\rangle = \sum_{\pm} \int \frac{d^3p}{(2\pi)^3} \theta[E_F - E_{\pm}(P)] \left[\frac{P_o - E_{\pm}(\underset{\sim}{P})}{D(P)} \mathrm{Tr}(\Gamma N(P))\right]_{P_o = E_{\pm}(\underset{\sim}{P})} \quad (7)$$

The integration is over the momentum region defined for each integral, $\pm$, by

$$E_{\pm}(\underset{\sim}{P}) = E_F \quad (8)$$

where E_F denotes the Fermi energy. From the defining equation for the eigenvalues, (1), it is evident that the Fermi surface is not a sphere but that it has axial symmetry along the direction $\underset{\sim}{K}$. In general this surface does not have reflection symmetry through the plane $P_{\parallel} = \underset{\sim}{P}\cdot\underset{\sim}{K}/|\underset{\sim}{K}| = 0$, except when $K_o = 0$, which corresponds to symmetric nuclear matter.[1] Because of this symmetry, integrals over odd powers of components of momentum perpendicular to $\underset{\sim}{K}$ vanish. In symmetric matter, $(K_o = 0)$, an integral over an odd power of $P_{\parallel}$ also vanishes.

According to (7) we need to evaluate certain traces of operators that appear in our source currents with the numerator, N, of S given by (6).

These traces can be evaluated using the properties of the Dirac matrices γ and the isospin matrices τ. See especially ref. 8 for some useful theorems. The results are

i) Scalar,

$$\mathrm{Tr}(N) = 8m^*[(PP) - \varepsilon_o^2 - 2(g_\pi \bar{\pi})^2(KK)] \quad (9)$$

ii) Vector,

$$\mathrm{Tr}(\gamma_\mu N) = 4[2((PP) - \varepsilon_o^2)P_\mu - (1 + (2g_\pi\bar{\pi})^2)(PK)K_\mu] \quad (10)$$

iii) Pion,

$$\mathrm{Tr}(\gamma_5 \gamma_\mu \tau_i) = -4g_\pi\bar{\pi}[2((PP) - \varepsilon_o^2 + 2m^{*2})K_\mu - 4(PK)P_\mu](\underset{\sim}{v} \times \underset{\sim}{u})_i \quad (11)$$

iv) Rho,

$$\mathrm{Tr}(\gamma_\mu \, \tau_i) = 4[((PP) - \varepsilon_o^2) \, K_\mu - 2(PK) \, P_\mu] \, v_i \tag{12}$$

These results when used in (7) provide the explicit integral expressions for the source currents appearing in the field equations (III-15). Notice that for our choice of condensate (III-3,4) only the τ_3 and τ_2 components, respectively, of the traces iii and iv survive.

The above method of calculating ground state expectation values is very general, perhaps more so than is necessary in the mean field approximation to the ground state. For example, in the preceding section, we showed that certain results such as VI-(20),(22),(27) could be obtained directly through the eigenvalue equation VI-(19). This alternative direct approach, which is suitable for mean field theories, will be elaborated elsewhere.

VIII. Summary of the Problem

We are now in a position to state precisely how to calculate the properties of matter that is described by our Lagrangian (II-15). The quantities that one must solve for, at given charge and baryon densities, are the mean meson fields, the pion frequency and wave number, and the Fermi energy of the quasi nucleons, σ, $\overline{\pi}$, ω_0, ρ_0, k_0, $|\underset{\sim}{k}|$, E_F. We have seen already that the space-like part of ω^μ and ρ^μ vanish. The above quantities are simultaneous solutions to seven equations. Four are the meson field equations (III-15). The remaining ones are

$$J_3^0 = \overline{\pi}^2 K^0 + \langle\overline{\Psi}_V\, \gamma^0\, (\tfrac{1}{2}\,\tau_3 + g_\pi \overline{\pi}\, \gamma_5\, \tau_2)\, \psi_V\rangle = q - \tfrac{1}{2}\,\rho_B \tag{1}$$

$$\underset{\sim}{J}_3 = \overline{\pi}^2 \underset{\sim}{K} + \langle\overline{\Psi}_V\, \underset{\sim}{\gamma}\, (\tfrac{1}{2}\,\tau_3 + g_\pi\, \overline{\pi}\, \gamma_5\, \tau_2)\, \psi_V\rangle = 0 \tag{2}$$

$$2 \sum_{\pm} \int \frac{d^3p}{(2\pi)^3}\, \theta[E_F - E_\pm(\underset{\sim}{P})] = \rho_B \tag{3}$$

(where $\theta(x) = 1$ if $x > 0$ and vanishes otherwise). The first two quantities are the time and space-like components of one of the conserved isospin four-currents $(J_{30}, \underset{\sim}{J}_3)$, corresponding to the third isospin axis. The other two isospin currents vanish identically for our condensates (III-3,4,10), as can be verified by referring to (II-15b) and the trace results (VII-10-11). The same trace results tell us that $\underset{\sim}{J}_3$ is in the direction of $\underset{\sim}{K}$, so that (2) constitutes a single equation. Recall that (1) and (2) follow from the condition that the pions should condense in the lowest energy mode of the medium available, defined by

$$\frac{\delta\overline{H}}{\delta k_0} = 0, \quad \frac{\delta\overline{H}}{\delta\underset{\sim}{k}} = 0 \tag{4}$$

Having established in section VI that $|k_0| = \mu_\pi$, the first of these equations assures chemical equilibrium among protons, neutrons, and pions, under the given constraints of charge and baryon density.

The equation (3) simply defines the Fermi energy E_F in terms of the baryon density and follows from (VI-20).

When the seven quantities mentioned above have been determined at chosen electric charge and baryon density, then the energy density and pressure can be calculated from the expressions given in section VI.

In the case that this theory is applied to neutron stars, then electrons also must be added to the system. They may be added as free Dirac particles since their interaction with the hadrons is only through the electro-weak interaction. Neutron stars are charge neutral; otherwise the long-range Coulomb interaction would counterbalance gravity. They are also in chemical equilibrium under $n \leftrightarrow p + e^-$. Through the condition on the charge, the electron Fermi energy becomes a function of the seven quantities listed above. Minimization of the total energy, including that of the electrons, through the first condition in (4) ensures chemical equilibrium and readily shows that $\mu_e = \mu_\pi = -k_0$.

A numerical application of the theory to isospin asymmetric matter and neutron stars will be presented elsewhere.*

Acknowledgements

NKG is deeply indebted to G. Baym from whom he learned techniques that were usefully applied in section VI. We are also indebted to P. Hecking for numerous discussions which helped to clarify the formulation. One of us (BB) wishes to acknowledge the warm hospitality of the Nuclear Theory Group of the Lawrence Berkeley Laboratory. This work was supported by the Director, Office of Energy Research, Division of Nuclear Physics of the Office of High Energy and Nuclear Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098.

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